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RESEARCH ARTICLE

Inherent Approaches to Solving Exponential Diophantine Equations involving Disarium Numbers of order 2 to 4.

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Abstract

Exponential Diophantine equations, which involve integer solutions to equations containing variables in exponents, represent a significant area of study in number theory. In this article, we explore, we introduce three distinct exponential Diophantine equations. By incorporating insights from the Catalan Conjecture and utilizing Disarium numbers of orders 2, 3, and 4, we establish non-negative integer solutions to these equations. A series of numerical examples is also presented to validate the proposed method and demonstrate how the equations can be effectively solved.

Keywords: Exponential Diophantine equations, Disarium Numbers, Integer solutions, Catalan's conjecture.

Introduction

Number theory is a branch of pure mathematics devoted to the study of the properties and relationships of integers. It encompasses a wide range of topics, including the distribution of prime numbers, Diophantine equations, modular arithmetic, and arithmetic functions. This field continues to evolve with the development of sophisticated techniques from algebraic number theory, analytic methods, and computational approaches, highlighting its central role in both theoretical mathematics and practical applications.

Diophantine equations arise when one or more variables appear as exponents. These are known as exponential Diophantine equations, and they fall under a broader category of Diophantine equations, which includes polynomial Diophantine equations, Pell's equation, infinite Diophantine equations, and others. The concept of History of Theory of Numbers and Diophantine Analysis Carmichael R. D in 1959. E. Catalan was the introduce the concept of exponential Diophantine equation for Note extradite d'une letter addressee a lediteur and The majority of the equations are solved using methods such as Catalan's conjecture or even trial and error. For various problems and ideas, one may refer has been studied exponential Diophantine equation. Janaki G and C. Saranya analyzed certain concepts that Exponential Diophantine Equation Involving Jarasandha Numbers, Ramanujan Prime Numbers of Order 2. Kannan J and Somanath M in suggested the On the Exponential Diophantine Equation Related to the Powers in 2024 and 2025.

This study presents three novel exponential Diophantine equations and investigates their non-negative integer solutions. Drawing on insights from Catalan's Conjecture and employing Disarium numbers of orders 2, 3, and 4, we develop solution strategies tailored to each equation

Method of Analysis:

Theorem: 1

For the Diophantine equation $D_s^x + (D_s - 1)^{2y} 2^{-2y} = z^2$ where is the Disarium number of orders two, then the following conditions are satisfied.

- If then the solution does not exist.
- If then there is no integer solution.
- (iii)If and there is a non-negative integral solution exist.

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Proof:

Let , and be a non-negative integer such that

$$D_s^x + (D_s - 1)^{2y} 2^{-2y} = z^2 \qquad (1)$$

We will analyse the solution of our considered in three cases.

Case (i)

If then becomes $z^2 - 1 = (D_s - 1)^{2y} 2^{-2y}$

Let $z - 1 = 2^u (D_s - 1)^w$ (2) then $z + 1 = 2^{-2y-u} (D_s - 1)^{2y-w}$ (3)

Subtracting equation (2) from (3), we get

$$2 = 2^u (D_s - 1)^w [2^{-2y-2u} (D_s - 1)^{2y-2w} - 1]$$

This implies that and then $(D_s - 1)^{2y} = 2^{2y+3}$ this is impossible.

Hence the Diophantine equation $D_s^x + 2^{-2y}(D_s - 1)^{2y} = z^2$ has no integer solution for the case .

Case (ii)

If then becomes

Let then we obtain

Let then this is also not possible.

Therefore, when there is no solution exist for equation .

Case (iii)

Consider the equation $z^2 - 2^{-2y}(D_s - 1)^{2y} = D_s^x$

Let $z - 2^{-y}(D_s - 1)^y = D_s^\sigma$ then

$$z + 2^{-y}(D_s - 1)^y = D_s^{x-\sigma}$$

From above two equations to solve them,

$$2[2^{-y}(D_s - 1)^y] = D_s^{x-\sigma} - D_s^\sigma$$

Let and we obtain, $D_s^x - 2^{1-y}(D_s - 1)^y = 1$

If and then $z - 2^{-1}(D_s - 1) = D_s^0$

This implies that, $z = \frac{D_s + 1}{2}$

Hence the Integral Solutions to the Diophantine equation $D_s^x + 2^{-2y}(D_s - 1)^{2y} = z^2$ is $\left(1, 1, \frac{D_s + 1}{2}\right)$

Theorem 2:

The Diophantine equation $(4D_s)^x + (D_s - 1)^{2y} = z^2$ where is the Disarium number of orders three. The following characteristics are stipulated.

- If and
- If then there is no integer solution for the Diophantine equation.
- If and then there exist an integer solution.

Proof:

Let us consider the Diophantine equation

$$2^{2x} D_s^x + (D_s - 1)^{2y} = z^2 \qquad (4)$$

Case (i)

If then equation (4) becomes,

$$z^2 - 1 = (D_s - 1)^{2y}$$

Let $z - 1 = (D_s - 1)^u$ (5)

And $z + 1 = (D_s - 1)^{2y-u}$ (6)

Subtraction equation (5) from (6) us get,

$$2 = (D_s - 1)^{2y-u} - (D_s - 1)^u$$
$$\Rightarrow (D_s - 1)^u \left[(D_s - 1)^{2y-2u} - 1\right]$$

This implies, and then $(D_s - 1)^{2y} = 2$ this is not possible.

So that there is no solution for (4) whenever .

Case (ii)

If then equation (4) becomes,

(ie)

Let and

then we obtain

This implies

⇒ ; then

Then the reduced equation this is impossible.

Therefore, no solution for the case in the Diophantine equation (4).

Case (iii)

Consider the equation (4) is $z^2 - (D_s - 1)^{2y} = 2^{2x} D_s^x$

Let $z - (D_s - 1)^y = 2^u D_s^\sigma$ then $z + (D_s - 1)^y = 2^{2-u} D_s^{x-\sigma}$

then we obtain, $2(D_s - 1)^y = 2^u D_s^\sigma \left[2^{2-2u} D_s^{x-2\sigma} - 1\right]$ this implies that, and then $(D_s - 1)^y = 2^{2-2} D_s^x - 1$.

If , then

Hence, the number of non-negative integral solutions of the Diophantine equation (4) is $(1, 1, D_s + 1)$

Theorem:3

$(1, 1, D_s + 2)$ is the solution of the exponential diophantine equation $[2^2(D_s + 1)]^x + D_s^{2y} = z^2$ (7), where x, y and z are non-negative integers.

Proof:

Case (i)

Suppose then becomes

Let where is non-negative integer such that we obtain these two assumptions

Then and this is not possible for .

Casewt (ii)

If then we get $z^2 - 1 = 2^{2x}(D_s + 1)^x$

if we let $z - 1 = 2^u (D_s + 1)^v$

where, and are non-negative integer, so that

Table 1

S. No	*Disarium number*	*Exponential Diophantine equation*	*Integer solutions*
	89	$89^x + 1936^y = z$	$(1,1,45)$
	135	$540^x + 134^y = z$	$(1,1,136)$
	175	$700^x + 174^y = z$	$(1,1,176)$
	518	$2072^x + 517^y = z$	$(1,1,519)$
	598	$2392^x + 597^y = z$	$(1,1,599)$
	1306	$5228^x + 1306^y = z$	$(1,1,1308)$
	1676	$6708^x + 1676^y = z$	$(1,1,1678)$
	2427	$9712^x + 2427^y = z$	$(1,1,2429)$

$$z+1 = 2^{2x-u}\left(D_s+1\right)^{x-v}$$

using the above two considerations we get

$$2 = 2^u\left(D_s+1\right)^v[2^{2x-2u}\left(D_s+1\right)^{x-2v} - 1]$$

Then ; and $2^{2x}(D_s+1)^x = 2^3$ this is not possible for positive values of and values of .

Case (iii)

Consider the equation $z^2 - D_s^{2y} = 2^2(D_s+1)^x$
Let $z - D_s^y = 2^u(D_s+1)^\sigma$ and $z + D_s^y = 2^u\left(D_s+1\right)^{x-\sigma}$

$$2\left(D_s^y\right) = 2^u\left(D_s+1\right)^\sigma[2^{2-2u}\left(D_s+1\right)^{x-2\sigma} - 1]$$

then , and $D_s^y = 2(D_s+1)^x - 1$ then .
Hence $(1,1,D_s+2)$ is solution of the exponential Diophantine equation .

Illustration

Few numerical solutions for the choice of satisfying the exponential Diophantine equation and their solutions are illustrated below: -

Conclusion

In this paper, we have presented non-zero distinct integer solutions to exponential Diophantine equations involving Disarium numbers. Furthermore, we introduced three distinct forms of such equations. Upon detailed examination, we found that none of these equations have solutions when and , while integer solutions exist for and in the third case. As a direction for future work, similar equations could be investigated by exploring alternative numerical combinations or by extending the analysis to other classes of numbers.

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References

Carmichael R. D., (1959). History of Theory of Numbers and Diophantine Analysis, Dover Publication, New York.

Catalan, E. (1844) Note Extraite Dune Lettre Adressee a Lediteur. Journal für die Reine und Angewandte Mathematik, 27, 192. http://dx.doi.org/10.1515/crll.1844.27.192

Hua L. K, (1982).Introduction to the Theory of Numbers, Springer-Verlag, Berlin-New York

Ivan Niven, HerbertS. Zuckerman and Hugh L. Montgomery (2004). An Introduction to the Theory of Numbers, John Wiley and Sons Inc, New York.

Janaki G and Sangeetha P (2025). Non-Negative Solution of the Exponential Diophantine Equation withThree Unknowns Including Prime and Disarium Numbers, Indian Journal of Natural Sciences, Vol.15 / Issue 88 International Bimonthly (Print) –Open Access ISSN: 0976 – 0997.

Nagell. T (1981). "Introduction to Number theory", Chelsea publishing company, New York.

Somchit Chotchaisthit (2012). On the diophantine equation where is a prime number, American J. Math. Sci., vol-1, issue 1.

Saranya C and Janaki G, 2019. Solution of Exponential Diophantine Equation Involving Jarasandha Numbers, Advances and Applications in Mathematical Sciences, Volume 18, Issue 12, October 2019, Pages 1625-1629, Mili Publications.

Saranya P and Janaki G, (2017). On the exponential diophantine equation $36^x + {}^y = z$ International Research Journal of Engineering and Technology 4(11) ,1042-1044.

Saranya C, (2024).Solutions of Pell's Equation and Exponential Diophantine Equation Utilizing Ramanujan Prime Numbers of Order 2, Ganita, Vol.74(2), 455-460.

Kannan J, Kaleeswari K& Mahalakshmi, M. (2024). On Exponential Diophantine Equations Involving Mersenne Primes. Journal of the Calcutta Mathematical Society, 20(1), 81-86 (2024).

Somanath, M., Bindu, V. A., & Das, R. (2025). On the Exponential Diophantine Equation Related to the Powers. Mathematics of Intelligent Computing and Data Science: ICMICDS-2022, Kochi, India, September 15–17, 484, 165.